

INFINITELY MANY PERIODIC ORBITS IN NON-EXACT OSCILLATING MAGNETIC FIELDS ON SURFACES WITH GENUS AT LEAST TWO FOR ALMOST EVERY LOW ENERGY LEVEL

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ABSTRACT. In this paper we consider oscillating non-exact magnetic fields on surfaces with genus at least two and show that for almost every energy level k below a certain value $\tau_+^*(g, \sigma)$ less than or equal to the *Mañé critical value of the universal cover* there are infinitely many closed magnetic geodesics with energy k .

1. INTRODUCTION

Let (M, g) denote a closed connected orientable Riemannian surface with genus at least two. Let $\sigma \in \Omega^2(M)$ be a non-exact 2-form, let ω_g denote the standard symplectic form on TM obtained by pulling back the canonical symplectic form $dp \wedge dq$ on T^*M via the Riemannian metric and let

$$\omega := \omega_g + \pi^* \sigma$$

be the *twisted symplectic form* determined by the pair (g, σ) . Let $E : TM \rightarrow \mathbb{R}$ denote the energy Hamiltonian

$$E(q, v) = \frac{1}{2} \|v\|_q^2$$

and $\varphi_t : TM \rightarrow TM$ represent the flow of the symplectic gradient of E with respect to ω . We call φ_t the *magnetic flow* of the pair (g, σ) . The reason for this terminology is that φ_t can be thought of as modeling the motion of a particle of unit mass and charge under the effect of a magnetic field represented by the 2-form σ .

The aim of this paper is to generalize the main theorem in [AMMP14] to the non-exact oscillating case, that is to prove the existence of infinitely many closed periodic orbits on almost every low energy level without any additional non-degeneracy assumption (such an assumption is needed in [AMP13] for the exact case). More precisely, to any magnetic pair (g, σ) we can associate the energy value $\tau_+^*(g, \sigma)$, which is strictly positive if and only if the 2-form σ changes sign (see Lemma 6.2). A 2-form σ of this kind is called *oscillating*. Then the following theorem holds.

Theorem 1.1. *Let (M, g) be a closed orientable Riemannian surface of genus at least two and let $\sigma \in \Omega^2(M)$ be a non-exact oscillating 2-form. For almost every $k \in (0, \tau_+^*(g, \sigma))$ there are infinitely many geometrically distinct closed magnetic geodesics with energy k .*

In [AMMP14] the same result has been shown when the 2-form σ is exact. There the energy interval $(0, \tau_+^*(g, \sigma))$ is replaced by the interval $(0, c_u(L))$, where $c_u(L)$ denotes the Mañé critical value of the universal cover of the Lagrange function $L : TM \rightarrow \mathbb{R}$ given by

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$L(q, v) = \|v\|_q^2/2 + \lambda_q(v)$, where $\lambda \in \Omega^1(M)$ is a primitive of σ . The Lagrangian L gives rise to an associated free-period action functional

$$S_k : \Lambda \longrightarrow \mathbb{R}, \quad (x, T) \longmapsto T \int_0^1 \left[L\left(x(t), \frac{\dot{x}(t)}{T}\right) + k \right] dt,$$

on the space $\Lambda := H^1(\mathbb{T}, M) \times (0, +\infty)$. The interest in this functional relies on the fact that its critical points are exactly the closed magnetic geodesics with energy k . The properties of S_k are very different according to whether k is supercritical or subcritical (for a beautiful survey, see [Abb13]).

For $k > c_u(L)$, there exists a global minimizer in every non-trivial free homotopy class since S_k is bounded from below on a fixed free homotopy class and it satisfies the Palais-Smale condition [CIPP00].

For $k < c_u(L)$, the functional is not bounded from below on each free homotopy class and there are examples where the Palais-Smale property fails. However, following Contreras' general result on autonomous Tonelli Lagrangian systems [Con06], in the subcritical range S_k has a mountain-pass geometry on the space of contractible loops. The two valleys are represented by the constant loops and by the loops with negative action. Moreover, for the specific case of exact magnetic flows on surfaces the results of Taimanov [Tai92a, Tai92b, Tai93] and independently of Contreras, Macarini and G. Paternain [CMP04] imply that, for k in this range, there exists a local minimizer α_k for S_k . If we suppose further that the minimizer is strict, then S_k has a mountain-pass geometry for every iterate α_k^n of α_k , since α_k^n is still a strict local minimizer. For every $n \geq 1$ the two valleys are a small neighborhood of α_k^n and the set of loops in the same free component of α_k^n with action less than $S_k(\alpha_k^n)$. Therefore, critical points of mountain-pass type do exist provided one can prove the convergence of the Palais-Smale sequences associated to the minimax values. An argument originally due to Struwe [Str90] (and used already for the free-period action functional by Contreras [Con06]) ensures such convergence on almost every energy level below $c_u(L)$ exploiting the fact that the functional S_k is monotone in k (see [AMMP14]). Combining this result with a study of the behaviour of the index of a critical point under iteration, the analogue of Theorem 1.1 for exact magnetic flows on subcritical energy levels has been proved in [AMMP14].

In the present paper we aim to prove Theorem 1.1 exactly by generalizing the scheme of proof carried out in [AMMP14] to our setting. The first difficulty is that in the non-exact case the Lagrangian action functional is not available anymore, because the closed 2-form σ does not admit a primitive. However, we can define a closed 1-form dS_k on Λ , whose vanishing points still correspond to magnetic geodesics. In general, the direct study of the vanishing set of the 1-form would require to apply the methods of Novikov theory for multivalued functionals to this infinite-dimensional setting, but no result has been obtained so far by pursuing this strategy.

The main idea to overcome this problem in our case is due to Will Merry [Mer10]. He shows that, if the integral of σ vanishes on every 2-torus $\mathbb{T}^2 \rightarrow M$, the closed 1-form dS_k is actually exact on Λ and a primitive $S_k : \Lambda \rightarrow \mathbb{R}$ can be explicitly written. Merry proved that the integrals of σ on 2-tori vanish provided the *Mañé critical value of the universal cover* $c(g, \sigma) \in \mathbb{R} \cup \{+\infty\}$ associated to the magnetic pair (g, σ) is finite [Mer10, Lemma 2.2]. Since on surfaces of genus at least two the Mañé critical value is always finite (see Lemma 6.1), we have the existence of the functional S_k in our case. We recall its basic properties in relation to the Palais-Smale condition in Section 3.

Moreover, S_k has locally (in particular near a critical point) the same structure as a Lagrangian action functional (with a primitive ϑ not defined on the whole M). Therefore, the local theory is the same as in the exact case: iterates of (strict) local minimizers are still

(strict) local minimizers (Lemma 6.5) and the Morse index of the critical points satisfies the same iteration properties as described in [AMP13, Section 1] and in [AMMP14]. In particular, as shown in [AMMP14] for the exact case, a sufficiently high iterate of a periodic orbit cannot be a mountain pass critical point of S_k (see Section 5 for further details).

We restrict our study to energy values below $\tau_+^*(g, \sigma)$ since, in this range, S_k is not bounded from below and there exists a local minimizer α_k of S_k by [Tai93] or [CMP04] (see Section 6). In Section 8 we will use the existence of this local minimizer and its iterates to construct a sequence of minimax values, which allows us to prove Theorem 1.1.

We shall mention that in [GGM14] it has been proved that, for surfaces of genus at least two with a nowhere-vanishing magnetic field, the twisted geodesic flow has infinitely many periodic orbits on every low energy level. There the fact that for nowhere-vanishing magnetic fields the energy levels are of contact type for low energies was essentially used. Therefore, this kind of result cannot be applied here, since as we will see in Proposition 7.1, low energy levels are never of contact type for oscillating 2-forms.

Finally, we briefly point out where in our argument it is crucially needed that the base manifold M has dimension two.

First, the existence of a local minimizer for S_k is so far only known when M is a surface and relies essentially on the fact that in dimension two, one can define an analogue of the functional S_k called \mathcal{T}_k on the space of oriented multi-curves, that are the boundary of surfaces embedded in M (see Section 6). Moreover, the critical points of the new functional are still multi-curves whose components are magnetic geodesics with energy k . Although S_k is not bounded from below on the connected components of the free loop-space, the functional \mathcal{T}_k is and one might try to apply the direct method of the calculus of variation to find its critical points. The space of embedded multi-curves is however not the best to work with and that is the reason why Taimanov in [Tai92a, Tai92b, Tai93] considered the restriction of \mathcal{T}_k to the space of the so-called *films*, whilst Contreras, Macarini and Paternain in [CMP04] took a suitable extension of \mathcal{T}_k to the space of integral 2-currents.

Second, the fact that (strict) local minimizers for S_k remain (strict) local minimizers also when iterated is in general true only on orientable surfaces (however, one can prove Theorem 1.1 also for non-orientable surfaces by working on the orientation double covering). The interested reader may look at [Hed32] for a counterexample in dimension greater than two and at [KH95, Example 9.7.1] for a counterexample in the non-orientable case.

These are two of the reasons why in dimension bigger than two much less is known about the existence of closed magnetic geodesics. One of the most general results in higher dimension is the existence of one contractible periodic orbit on *almost* every low energy level if $\sigma \neq 0$ [Sch06]. Existence for *every* low energy can be obtained by requiring σ to be symplectic [GG09, Ush09]. Finally, multiplicity results hold if σ is a Kähler form [Ker99]. We also refer to the beautiful survey [Abb13], or to [Con06], for a discussion of the dynamics of exact magnetic forms in any dimension within the abstract framework of Tonelli Lagrangians.

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2. PRELIMINARIES

The contents of Sections 2–5 hold more generally for any closed connected Riemannian manifold (M, g) . Set

$$\Lambda := H^1(\mathbb{T}, M) \times (0, +\infty).$$

We denote by $\widetilde{\Lambda}$ the same object on the universal cover \widetilde{M} and by $p : \widetilde{\Lambda} \rightarrow \Lambda$ the projection map. The space Λ represents the set of loops with arbitrary period through the identification

$$(x, T) \in \Lambda \iff \gamma : [0, T] \rightarrow M, \quad \gamma(t) := x(t/T).$$

For every free homotopy class of loops $\nu \in [\mathbb{T}, M]$, let Λ_ν be the connected component of Λ given by loops in the class ν and denote by Λ_0 the connected component of contractible loops. We endow Λ with the structure of a Hilbert manifold given by the product metric. The space Λ carries a natural \mathbb{N} -action given by iteration

$$(1) \quad \psi : \mathbb{N} \times \Lambda \longrightarrow \Lambda, \quad (n, (x, T)) \longmapsto \psi(n, (x, T)) = \psi^n(x, T) := (x^n, nT),$$

where $x^n(t) := x(nt)$. In particular, for every class ν , there exists a unique class ν^n such that $\psi^n(\Lambda_\nu) \subset \Lambda_{\nu^n}$ and $\psi^n : \Lambda_\nu \rightarrow \Lambda_{\nu^n}$ is an embedding of Hilbert manifolds.

Let $\sigma \in \Omega^2(M)$ be a closed 2-form and let $\tilde{\sigma}$ be its pull-back to \widetilde{M} . We assume σ to be *weakly exact*, in the sense that $\tilde{\sigma}$ is exact (this is equivalent to requiring that $\sigma|_{\pi_2(M)} = 0$). Denote by $\varphi_t : TM \rightarrow TM$ the magnetic flow of the pair (g, σ) , i.e. the flow of the symplectic gradient of the Hamiltonian given by the kinetic energy with respect to the twisted symplectic form

$$\omega := \omega_g + \pi^* \sigma,$$

where ω_g is the pull-back on TM of the standard symplectic form on T^*M via the Riemannian metric and $\pi : TM \rightarrow M$ is the tangent bundle. We are interested in finding periodic orbits for φ_t . We call γ a *closed magnetic geodesic* of the pair (g, σ) if it is the projection under π of a periodic orbit of φ_t .

Now, fix a primitive ϑ of $\tilde{\sigma}$ and consider the Lagrangian

$$L : T\widetilde{M} \longrightarrow \mathbb{R}, \quad L(q, v) := \frac{1}{2} \|v\|_q^2 + \vartheta_q(v).$$

The Euler-Lagrange flow of L is precisely the lifted flow $\tilde{\varphi}_t : T\widetilde{M} \rightarrow T\widetilde{M}$ of the magnetic flow $\varphi_t : TM \rightarrow TM$ (see for instance [CI99]). Consider the action functional $\tilde{S}_k : \widetilde{\Lambda} \rightarrow \mathbb{R}$ defined by

$$\tilde{S}_k(\tilde{x}, T) := T \int_{\mathbb{T}} \left[L\left(\tilde{x}(t), \frac{\dot{\tilde{x}}(t)}{T}\right) + k \right] dt$$

and define the *Mañé critical value* of the pair (g, σ) as

$$c(g, \sigma) := \sup \left\{ k \in \mathbb{R} \mid \exists \tilde{\gamma} \in \widetilde{\Lambda} \text{ s.t. } \tilde{S}_k(\tilde{\gamma}) < 0 \right\} \in \mathbb{R} \cup \{+\infty\},$$

where as usual we make the identification $\tilde{\gamma} = (\tilde{x}, T)$. The next lemma implies that $c(g, \sigma)$ is finite if and only if the lift $\tilde{\sigma}$ of σ to the universal cover admits a bounded primitive (see [BP02] for a proof).

Lemma 2.1. *We can alternatively define $c(g, \sigma)$ as follows*

$$(2) \quad c(g, \sigma) = \inf_{u \in C^\infty(\widetilde{M})} \sup_{q \in \widetilde{M}} \frac{1}{2} \|d_q u - \vartheta_q\|^2.$$

Following [Mer10] we now introduce a second functional $S_k : \Lambda \rightarrow \mathbb{R}$ when $c(g, \sigma) < \infty$. The key observation is the following

Lemma 2.2. *If $c(g, \sigma) < \infty$, then for any $f : \mathbb{T}^2 \rightarrow M$ smooth, $f^* \sigma$ is exact.*

We refer to [Mer10] and references therein, in particular [Pat06], for a proof and a discussion of this result.

For every $\nu \in [\mathbb{T}, M]$ pick a reference loop x_ν in the connected component of $H^1(\mathbb{T}, M)$ corresponding to ν . Given any other loop $x \in H^1(\mathbb{T}, M)$ in the same connected component, let $C(x)$ denote a cylinder connecting x_ν to x and define $S_k : \Lambda_\nu \rightarrow \mathbb{R}$ by

$$(3) \quad S_k(x, T) := T \int_0^1 \left(\frac{1}{2T^2} |\dot{x}(t)|^2 + k \right) dt + \int_{C(x)} \sigma.$$

This is well defined because the value of the integral of σ over $C(x)$ is independent of the choice of the cylinder. Indeed, if $C'(x)$ is another cylinder with the same boundary, then

$$\mathbb{T}^2(x) := C(x) \cup \overline{C'(x)}$$

is a torus (where $\overline{C'(x)}$ denotes the cylinder $C'(x)$ taken with opposite orientation) and hence the integral of σ over $\mathbb{T}^2(x)$ is zero by Lemma 2.2. Observe that S_k depends on the choice of the reference loop. However, if we change x_ν to x'_ν , the functional only changes by the addition of a constant (the integral of σ over a cylinder connecting x_ν and x'_ν) and, hence, the geometric properties of S_k are unchanged.

When $\nu = 0$ is the class of contractible loops, we make the natural choice of taking as reference loop x_0 a fixed constant loop. Then, if $\gamma = (x, T) \in \Lambda_0$, the cylinder $C(x)$ is just a capping disc for x . As a consequence, if $\gamma' \in \Lambda$ is such that $\gamma'(0) = \gamma(0)$, then

$$(4) \quad S_k(\gamma * \gamma') = S_k(\gamma) + S_k(\gamma'),$$

where $*$ denotes the concatenation of loops.

Moreover, the functional $S_k|_{\Lambda_0}$ is related with \tilde{S}_k by the formula

$$S_k(p(\tilde{x}, T)) = \tilde{S}_k(\tilde{x}, T),$$

so that $S_k|_{\Lambda_0}$ is unbounded from below if $k < c(g, \sigma)$. By Equation (4), we get then the following result.

Lemma 2.3. *If $k < c(g, \sigma)$, then $S_k|_{\Lambda_\nu}$ is unbounded from below for every $\nu \in [\mathbb{T}, M]$.*

It is a straightforward computation to show that the functional S_k is of class C^2 (actually smooth). We refer to [Mer10] for the details.

The following lemma, contained in [Mer10, Corollary 2.3], explains why we are interested in the functional S_k .

Lemma 2.4. *A loop $\gamma \in \Lambda$ is a critical point for S_k if and only if it is a magnetic geodesic with energy k .*

The main result known so far about the critical points of S_k in this generality is Theorem 1.1 of [Mer10], which we now state.

Theorem 2.5 (W. Merry, 2010). *Let (M, g) be a closed connected Riemannian manifold and let $\sigma \in \Omega^2(M)$ be a closed weakly exact 2-form. Let $c := c(g, \sigma) \in \mathbb{R} \cup \{+\infty\}$ denote the Mañé critical value and φ_t the magnetic flow defined by σ . Suppose in addition that $c < \infty$. Then:*

- (1) *if $k > c$, for each non trivial free-homotopy class $\nu \in [\mathbb{T}, M]$, there is a closed orbit of φ_t with energy k such that its projection to M belongs to ν ;*
- (2) *for a.e. $k \in (0, c)$, there is a closed orbit of φ_t with energy k such that its projection to M is contractible.*

We finish this section by describing explicitly the behaviour of S_k under iteration.

Lemma 2.6. *For every $\nu \in [\mathbb{T}, M]$ and $n \in \mathbb{N}$, there exists $b(\nu, n) \in \mathbb{R}$, depending on the reference loops $\{x_{\nu^n}\}_{n \in \mathbb{N}}$, such that*

$$(5) \quad S_k(\psi^n(\gamma)) = n \cdot S_k(\gamma) + b(\nu, n), \quad \forall \gamma \in \Lambda_\nu.$$

Choosing $x_{\nu^n} = x_\nu^n$ when ν has infinite order, or x_ν equal to a constant path when $\nu = 0$, yields $b(\nu, n) = 0$ for every $n \in \mathbb{N}$. In this case, S_k is \mathbb{N} -equivariant:

$$(6) \quad S_k(\psi^n(\gamma)) = n \cdot S_k(\gamma), \quad \forall \gamma \in \Lambda_\nu.$$

Proof. Let $C(x)$ be a cylinder connecting x_ν to x and denote by $C^n(x)$ the cylinder obtained by iterating each loop n times. We construct a cylinder $C(x^n)$ from x_{ν^n} to x^n as the concatenation of a cylinder $C(x_\nu^n)$ connecting x_{ν^n} with x_ν^n and the cylinder $C^n(x)$ connecting x_ν^n with x^n . We compute

$$\int_{C(x^n)} \sigma = \int_{C(x_\nu^n) \cup C^n(x)} \sigma = \int_{C(x_\nu^n)} \sigma + \int_{C^n(x)} \sigma = b(\nu, n) + n \int_{C(x)} \sigma,$$

where

$$b(\nu, n) := \int_{C(x_\nu^n)} \sigma.$$

Plugging this identity into the definition of $S_k(\gamma^n)$, we get

$$\begin{aligned} S_k(\gamma^n) &= \int_0^{nT} \left(\frac{1}{2} |\dot{\gamma}(t')|^2 + k \right) dt' + \int_{C(x^n)} \sigma \\ &= n \int_0^T \left(\frac{1}{2} |\dot{\gamma}(t')|^2 + k \right) dt' + b(\nu, n) + n \int_{C(x)} \sigma \\ &= n \cdot S_k(\gamma) + b(\nu, n). \end{aligned} \quad \square$$

3. THE PALAIS-SMALE CONDITION FOR S_k

In this section we recall the basic properties of the functional S_k . All the results we mention are contained and proved in [Mer10] and in [AMP13]. They hold for any closed Riemannian manifold M with $c(g, \sigma) < \infty$.

Definition 3.1. *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Riemannian Hilbert manifold and consider $S : \mathcal{H} \rightarrow \mathbb{R}$ a functional of class C^1 . We say that S satisfies the Palais-Smale condition, if every sequence $\{x_n\} \subseteq \mathcal{H}$ satisfying*

$$\|dS(x_n)\| \rightarrow 0, \quad \sup_{n \in \mathbb{N}} |S(x_n)| < \infty$$

admits a convergent subsequence. We say that S satisfies the Palais-Smale condition at level $\mu \in \mathbb{R}$, if every sequence $\{x_n\} \subseteq \mathcal{H}$ satisfying

$$\|dS(x_n)\| \rightarrow 0, \quad S(x_n) \rightarrow \mu$$

admits a convergence subsequence.

The first result we present states that the functional S_k satisfies locally the Palais-Smale condition. Palais-Smale sequences (x_n, T_n) with $\liminf T_n = 0$ are a possible source of non-compactness, but it turns out that they only occur at level zero.

Theorem 3.2. *Suppose $(x_n, T_n) \subseteq \Lambda$ is a Palais-Smale sequence with T_n bounded from above. Then the following hold:*

- (1) *If $\liminf T_n > 0$, then, passing to a subsequence if necessary, the sequence (x_n, T_n) is convergent in the H^1 -topology.*

(2) If $\liminf T_n = 0$, then, passing to a subsequence if necessary,

$$S_k(x_n, T_n) \longrightarrow 0.$$

We refer to [Mer10] for the proof. In the light of this result, we see that S_k satisfies the Palais-Smale condition provided Palais-Smale sequences have bounded period. A possible generalization of Proposition F in [Con06] would yield that if the energy level $E^{-1}(k)$ is of contact type, then the Palais-Smale condition holds. Unfortunately, for sufficiently low energies, $E^{-1}(k)$ is never of contact type when the form is oscillating, as we will see in Section 7. Hence, to overcome the possible lack of the Palais-Smale condition, in Chapter 9 we will use an argument due to Struwe [Str90] to construct Palais-Smale sequences with bounded period on almost all energy levels below $\tau_+^*(g, \sigma)$.

Recall that $\alpha = (x, T) \in \Lambda$ is said to be a *strict local minimizer* of S_k on Λ , if the \mathbb{T} -orbit of α

$$\mathbb{T} \cdot \alpha := \left\{ (x(\tau + \cdot), T) \mid \tau \in \mathbb{T} \right\}$$

has a neighborhood \mathcal{U} in Λ , such that

$$S_k(\gamma) > S_k(\alpha), \quad \forall \gamma \in \mathcal{U} \setminus \mathbb{T} \cdot \alpha.$$

The fact that the Palais-Smale condition holds locally allows to prove that the \mathbb{T} -orbit of a strict local minimizer α has neighborhoods on whose boundary the infimum of S_k is strictly larger than $S_k(\alpha)$. The proof given in [AMP13, Lemma 4.3] goes through without any change.

Lemma 3.3. *Let $\alpha = (x, T)$ be a strict local minimizer for S_k on Λ . If the neighborhood \mathcal{U} of $\mathbb{T} \cdot \alpha$ is sufficiently small then*

$$\inf_{\partial \mathcal{U}} S_k > S_k(\alpha).$$

The next result ensures that the flow of $-\nabla S_k$ is relatively complete on Λ' , the complement in Λ of the space of contractible loops Λ_0 . Since Λ is not complete we may not expect the flow of $-\nabla S_k$ to be positively complete. However, the second statement of the next lemma ensures that it is relatively complete in Λ_0 on any interval that does not contain zero.

Lemma 3.4. *Let $k > 0$ and let $\Lambda = \Lambda' \sqcup \Lambda_0$. The following two properties hold:*

- (1) *The sublevels $\{S_k \leq c\}$ are complete on Λ' . More precisely, if $(x_n, T_n) \subseteq \Lambda'$ is such that $T_n \rightarrow 0$, then $S_k(x_n, T_n) \rightarrow +\infty$.*
- (2) *If $[a, b] \subseteq \mathbb{R}$ is an interval such that $0 \notin [a, b]$, then the local flow of $-\nabla S_k$ is relatively complete on $\Lambda_0 \cap \{a \leq S_k \leq b\}$.*

The proof of the first statement follows directly from the proof of Lemma 3.2 in [Abb13] (there the case of the Lagrangian action functional is treated). We refer to [Mer10, Lemma 5.7] for the proof of the second statement.

4. THE ACTION FUNCTIONAL NEAR CRITICAL POINTS

Let ν be a free homotopy class which is either non-torsion or trivial. Suppose that $\gamma = (x, T)$ is a magnetic geodesic with energy k in the free homotopy class ν and take x and its iterates as the reference loops in the definition of S_k . Let

$$\pi_{x^n} : (x^n)^*(TM) \longrightarrow \mathbb{T}$$

be the pull-back bundle of $\pi : TM \rightarrow M$ under the map $x^n : \mathbb{T} \rightarrow M$. There exists an open neighborhood U of the zero-section of $x^*(TM)$ such that

$$\exp : U \longrightarrow M, \quad \exp(t, \xi) := \exp_{x(t)}(\xi)$$

is an immersion. Since $x(\mathbb{T}) \neq M$, by shrinking the open set U if necessary, we can assume that $V := \exp(U) \neq M$ and take a 1-form $\vartheta \in \Omega^1(V)$ such that $d\vartheta = \sigma$ and $\|\vartheta\| < \infty$.

For every $n \in \mathbb{N}$ let U_n be an open neighborhood of the zero section of $(x^n)^*(TM)$, such that $\exp(U_n) = \exp(U) = V$. Denote by $H^1(U_n)$ the space of H^1 -sections of π_{x^n} with image contained in U_n . Then

$$i_n : H^1(U_n) \times (0, +\infty) \longrightarrow \Lambda_{\nu^n}, \quad i_n(t, \xi(t), T) := (\exp(t, \xi(t)), T)$$

is a diffeomorphism with an open neighborhood of γ^n inside Λ_{ν^n} .

Take now (y, S) in this neighborhood and observe that there exists a cylinder $C(x^n, y)$ entirely contained in V . Therefore,

$$\int_{C(x^n, y)} \sigma = \int_{C(x^n, y)} d\vartheta = \int_0^1 y^* \vartheta - n \int_0^1 x^* \vartheta.$$

Hence, we can write

$$(7) \quad S_k(y, S) = \overline{S}_k(y, S) - n \int_0^1 x^* \vartheta,$$

where $\overline{S}_k : i_n(H^1(U_n) \times (0, +\infty)) \longrightarrow \mathbb{R}$ is defined as

$$(8) \quad \overline{S}_k(y, S) := S \int_0^1 \left[\frac{1}{2S^2} |\dot{y}(t)|^2 + k \right] dt + \int_0^1 y^* \vartheta.$$

From this it follows all the local results that hold for the case of exact magnetic fields can be used in our setting, as well. Proposition 1.1 of [AMMP14] applied in this context ensures that the mean index of a critical point γ of S_k vanishes if and only if the free-period Morse index vanishes for every $n \in \mathbb{N}$.

Lemma 4.1. *If γ is a critical point for S_k , then $\text{ind}(\gamma) = 0$ for every $n \in \mathbb{N}$ if and only if $\overline{\text{ind}}(\gamma) = 0$.*

When dealing with degenerate critical points the following lemma (originally due to Gromoll and Meyer) turns out to be useful. We refer to [AMMP14] for the proof.

Lemma 4.2. *Let $\gamma = (x, T)$ be a critical point of S_k which corresponds to a non-constant periodic orbit. Then, there is a partition $\mathbb{N} = \mathbb{N}_1 \cup \dots \cup \mathbb{N}_k$, integers $n_1 \in \mathbb{N}_1, \dots, n_k \in \mathbb{N}_k$, and $\nu_1, \dots, \nu_k \in \{0, \dots, 2d - 2\}$ with the following property: n_j divides all the integers in \mathbb{N}_j , and $\text{null}(\gamma^n) = \nu_j$ for all $n \in \mathbb{N}_j$.*

The following tubular neighborhood lemma (see [GM69, Section 3]), will be needed in the proof of Theorem 5.1.

Lemma 4.3. *Let γ be a critical point of the free-period action functional S_k such that, for some $n \in \mathbb{N}$, we have $\text{ind}(\gamma) = \text{ind}(\gamma^n)$ and $\text{null}(\gamma) = \text{null}(\gamma^n)$. Then, for every sufficiently small open neighborhood \mathcal{U} of $\mathbb{T} \cdot \gamma$ there is an open neighborhood \mathcal{V} of the submanifold $\psi^n(\mathcal{U})$ and a smooth map*

$$r : [0, 1] \times \mathcal{V} \longrightarrow \mathcal{V}, \quad (t, \beta) \mapsto r_t(\beta)$$

such that the following hold:

- (i) $r_0 = \text{id}$;
- (ii) $r_t|_{\psi^n(\mathcal{U})} = \text{id}$, for every $t \in [0, 1]$;
- (iii) $r_1(\mathcal{V}) = \psi^n(\mathcal{U})$;
- (iv) $\frac{d}{dt} S_k(r_t(\beta)) < 0$ for all $\beta \in \mathcal{V} \setminus \psi^n(\mathcal{U})$.

We end this section recalling the following result which states that any isolated critical circle of S_k admits a fundamental system of connected open neighborhoods \mathcal{U} such that the intersection of \mathcal{U} with the energy sublevel has finitely many connected components. Assuming, furthermore, that the critical point has free-period Morse index at least two, one gets that the intersection of such neighborhoods with the energy sublevel is non-empty and connected. We refer to [AMMP14, Lemma 2.5] for the proof.

Lemma 4.4. *Let $\mathbb{T} \cdot \gamma$ be an isolated critical circle of the free-period action functional S_k with critical value $c := S_k(\gamma)$. Then this circle admits a fundamental system of connected open neighborhoods \mathcal{U} such that*

$$\mathcal{U} \cap \{S_k < c\}$$

has finitely many connected components. Moreover, if $\text{ind}(\gamma) \geq 2$, for every such neighborhood the intersection

$$\mathcal{U} \cap \{S_k < c\}$$

is non-empty and connected.

5. ITERATION OF CRITICAL POINTS OF S_k

Here we use the results of Section 4 to prove an analogue of [AMMP14, Theorem 2.6]. Strictly speaking, this shows that critical circles of the functional S_k cannot be of mountain pass type if iterated sufficiently many times.

Theorem 5.1. *Let $\mathbb{T} \cdot \gamma$ be a critical circle of the free-period action functional S_k . Assume that all the iterates of γ belong to isolated critical circles of S_k . Then, for all $n \in \mathbb{N}$ large enough there exists an open neighborhood \mathcal{W} of $\mathbb{T} \cdot \gamma^n$ with the following property: if two points $\gamma_0, \gamma_1 \in \{S_k < S_k(\gamma^n)\}$ are contained in the same connected component of*

$$\{S_k < S_k(\gamma^n)\} \cup \mathcal{W},$$

then they are contained in the same connected component of $\{S_k < S_k(\gamma^n)\}$.

Proof. If $\overline{\text{ind}}(\gamma) > 0$, then for n large enough, $\text{ind}(\gamma) \geq 2$. Therefore the theorem follows directly from Lemma 4.4. Thus we are left with the case $\overline{\text{ind}}(\gamma) = 0$, which means, in virtue of Lemma 4.1, that $\text{ind}(\gamma^n) = 0$ for every $n \in \mathbb{N}$. Thanks to Lemma 4.2 we get a partition $\mathbb{N} = \mathbb{N}_1 \cup \dots \cup \mathbb{N}_h$ and integers $n_1 \in \mathbb{N}_1, \dots, n_h \in \mathbb{N}_h$, $\nu_1, \dots, \nu_h \in \{0, \dots, 2d-2\}$ with the property that if $n \in \mathbb{N}_j$, then n is a multiple of n_j and $\text{null}(\gamma^n) = \nu_j$. Without loss of generality let us suppose that $n \in \mathbb{N}_1$. By Lemma 4.4 there is a connected neighborhood \mathcal{U} of γ^{n_1} such that

$$\mathcal{U}^- := \mathcal{U} \cap \{S_k < S_k(\gamma^{n_1})\}$$

has finitely many connected components $\mathcal{U}_1^-, \dots, \mathcal{U}_r^-$. We can also assume that on \mathcal{U} the action functional can be written in the form $S_k = \overline{S}_k + c$, where \overline{S}_k is defined as in (8). For every $\alpha \in \{1, \dots, r\}$ choose a loop $\gamma_\alpha \in \mathcal{U}_\alpha^-$ and for each pair of distinct $\alpha, \beta \in \{1, \dots, r\}$ choose a path

$$\Theta_{\alpha\beta} : [0, 1] \longrightarrow \mathcal{U}$$

connecting γ_α and γ_β . By Bangert's technique of "pulling one loop at a time" (see [Ban80, pages 86–87] or [Abb13, page 421]) applied to $\Theta_{\alpha\beta}$ and the functional \overline{S}_k , for all sufficiently large multiples n of n_1 each path $\psi^{n/n_1} \circ \Theta_{\alpha\beta}$ consisting of iterated loops is homotopic with fixed endpoints to a path entirely contained in the sublevel set

$$\{\overline{S}_k < \overline{S}_k(\gamma^n)\} \subset \{S_k < S_k(\gamma^n)\}.$$

In other words, $\psi^{n/n_1}(\mathcal{U}^-)$ is contained in a path-connected component of

$$\{S_k < S_k(\gamma^n)\},$$

provided $n \in \mathbb{N}_1$ is sufficiently large. Consider such an $n \in \mathbb{N}_1$: applying Lemma 4.3 to γ^{n_1} and γ^n we get neighborhoods $\mathcal{U}' \subseteq \mathcal{U}$ of $\mathbb{T} \cdot \gamma^{n_1}$ and \mathcal{V}' of $\mathbb{T} \cdot \gamma^n$ and a corresponding deformation retraction $r : \mathcal{V}' \rightarrow \mathcal{V}'$ onto $\psi^{n/n_1}(\mathcal{U}')$ such that

$$\frac{d}{dt} S_k \circ r_t \leq 0.$$

Set now \mathcal{W} to be an open neighborhood of γ^n whose closure is contained in \mathcal{V}' . Let γ_0 and γ_1 in $\{S_k < S_k(\gamma^n)\}$ be as in the statement of the theorem and let

$$\Gamma : [0, 1] \longrightarrow \mathcal{W} \cup \{S_k < S_k(\gamma^n)\}$$

be a path joining them. Consider s_{\pm} the infimum and the supremum of the times $s \in [0, 1]$ such that $\Gamma(s) \in \mathcal{W}$. Then, $\Gamma(s_{\pm})$ belong to

$$\mathcal{V}' \cap \{S_k < S_k(\gamma^n)\}$$

and the paths $t \mapsto r_t(\Gamma(s_{\pm}))$ are entirely contained in $\{S_k < S_k(\gamma^n)\}$ by Lemma 4.3. Moreover $r_1(\Gamma(s_{\pm})) \in \psi^{n/n_1}(\mathcal{U}^-)$. By what we proved before it follows that $r_1(\Gamma(s_-))$ and $r_1(\Gamma(s_+))$ are in the same connected component of $\{S_k < S_k(\gamma^n)\}$. Thus, the same is true for γ_0 and γ_1 . \square

6. LOCAL MINIMIZERS ON SURFACES

From now on, we suppose that (M, g) is an orientable surface of genus at least two. Let $\sigma \in \Omega^2(M)$ be a non-exact 2-form on M and denote by φ_t the magnetic flow defined by σ . Changing the orientation of M if necessary, we may suppose that σ has positive integral over M . The following lemma shows that the Mañé critical value is finite in this case and, hence, the functional S_k introduced in Section 2 is well-defined on Λ .

Lemma 6.1. *If M has genus at least two, then $c(g, \sigma) < \infty$. Namely, σ admits a bounded primitive on the universal cover.*

Proof. Consider first the case of g_0 , the Riemannian metric with constant curvature -1 on M and σ_0 the associated area form. Take the model

$$\left\{ (x, y) \in \mathbb{R}^2 \mid y > 0 \right\}$$

for the universal cover \widetilde{M} and let $p : \widetilde{M} \rightarrow M$ be the covering map. Then,

$$p^*g_0 = \frac{dx^2 + dy^2}{y^2}, \quad p^*\sigma_0 = \frac{1}{y^2} dx \wedge dy.$$

Define $\widetilde{\beta}_0 := \frac{dx}{y}$ a primitive for $p^*\sigma_0$ and compute

$$\|\widetilde{\beta}_0\|_{p^*g_0} = \frac{1}{y} \|dx\|_{p^*g_0} = \frac{1}{y} y = 1.$$

Hence, $p^*\sigma_0$ admits a bounded primitive on \widetilde{M} .

If now (g, σ) is any pair defining a magnetic system, with $\int_M \sigma \neq 0$, we find a positive constant $A(g)$ and a 1-form $\beta_\sigma \in \Omega^1(M)$ with

$$\|v\|_g \geq A(g) \cdot \|v\|_{g_0}, \quad \sigma = \frac{\int_M \sigma}{2\pi\chi(M)} \sigma_0 + d\beta_\sigma.$$

If $\widetilde{\beta}_0$ is the primitive of σ_0 considered before, we have that

$$\frac{\int_M \sigma}{2\pi\chi(M)} \widetilde{\beta}_0 + p^*\beta_\sigma$$

is a primitive for σ and

$$\begin{aligned} \left\| \frac{\int_M \sigma}{2\pi\chi(M)} \tilde{\beta}_0 + p^* \beta_\sigma \right\|_{p^*g} &\leq \left| \frac{\int_M \sigma}{2\pi\chi(M)} \right| \cdot \left\| \tilde{\beta}_0 \right\|_{p^*g} + \|p^* \beta_\sigma\|_{p^*g} \\ &\leq \left| \frac{\int_M \sigma}{2\pi\chi(M)} \right| \cdot \left\| \frac{1}{A(g)} \tilde{\beta}_0 \right\|_{p^*g_0} + \|\beta_\sigma\|_g < \infty. \quad \square \end{aligned}$$

In the remainder of this section we prove that, if the energy value k is sufficiently small, then there exists a closed magnetic geodesic of energy k which is a local minimizer of S_k , provided that σ is a so-called *oscillating form*, i.e. that there is a point $x \in M$ such that $\sigma_x < 0$.

Let us start by defining the family of Taimanov functionals $\mathcal{T}_k : \mathcal{F}_+ \rightarrow \mathbb{R}$, where $k \in (0, +\infty)$ and \mathcal{F}_+ is the space of positively oriented (possibly with boundary and not necessarily connected) embedded surfaces in M (in [Tai92a, Tai92b, Tai93], Taimanov considered the so-called *films*):

$$(9) \quad \mathcal{T}_k(\Pi) := \sqrt{2k} \cdot l(\partial\Pi) + \int_\Pi \sigma,$$

where $l(\partial\Pi)$ denotes the length of the boundary of Π . We readily compute for future applications (notice that $\emptyset \in \mathcal{F}_+$)

$$(10) \quad \mathcal{T}_k(\emptyset) = 0, \quad \mathcal{T}_k(M) = \int_M \sigma > 0.$$

Observe that the family $\{\mathcal{T}_k\}$ is increasing in k and each \mathcal{T}_k is bounded from below since

$$\mathcal{T}_k(\Pi) \geq -\|\sigma\|_\infty \cdot \text{area}_g(M).$$

Define the value

$$(11) \quad \tau_+(M, g, \sigma) := \inf \{k \mid \inf \mathcal{T}_k \geq 0\} = \sup \{k \mid \inf \mathcal{T}_k < 0\}.$$

The functionals \mathcal{T}_k can be lifted to any finite cover $p' : M' \rightarrow M$ giving rise to the set of values $\tau_+(M', g, \sigma)$. We can define the *Taimanov critical value* as

$$(12) \quad \tau_+(g, \sigma) := \sup \left\{ \tau_+(M', g, \sigma) \mid p' : M' \rightarrow M \text{ finite cover} \right\}.$$

In [CMP04] it was shown that, when σ is exact, $\tau(g, \sigma) = c_0(g, \sigma)$, where $c_0(g, \sigma)$ denotes the Mañé critical value of the abelian cover of M (here we write τ instead of τ_+ , since in the exact case one does not have to struggle with the orientation of the embedded submanifolds). When σ is not exact, $\tau_+(g, \sigma)$ is positive exactly when σ is oscillating as the next lemma shows. In order to have a more precise representation of the Taimanov critical value, let us define the auxiliary value $c_+(g, \sigma)$

$$\begin{aligned} c_+(g, \sigma) &:= \inf \left\{ k \in \mathbb{R} \mid \forall \gamma \in \Gamma, \tilde{S}_k(\tilde{\gamma}) \geq 0 \right\} \\ &= \sup \left\{ k \in \mathbb{R} \mid \exists \gamma \in \Gamma \text{ s.t. } \tilde{S}_k(\tilde{\gamma}) < 0 \right\}, \end{aligned}$$

where Γ is the class of positively oriented simple loops on the universal cover of M . For the proof of the main theorem we need to work with energy levels below $\tau_+(g, \sigma)$ that are also smaller than $c(g, \sigma)$. Since we do not know if, in general, there is an inequality between these two quantities, we simply set

$$\tau_+^*(g, \sigma) := \min \left\{ \tau_+(g, \sigma), c(g, \sigma) \right\}.$$

Lemma 6.2. *Suppose $\sigma \in \Omega^2(M)$ is a non-exact 2-form on M with positive integral. Then $c_+(g, \sigma) > 0$ if and only if σ is oscillating. Furthermore,*

$$\tau_+^*(g, \sigma) \geq c_+(g, \sigma).$$

Proof. If σ is everywhere non-negative, then $\tilde{S}_k(\tilde{\gamma}) \geq 0$ for every $k > 0$ and every $\tilde{\gamma} \in \Gamma$. Conversely, assume that $\tilde{\sigma}$ is negative at some point \tilde{x} . Then there exists a small positively oriented disc \tilde{D} around \tilde{x} such that

$$\int_{\tilde{D}} \tilde{\sigma} < 0.$$

Given $k > 0$, we define $\tilde{y}_k : [0, l(\partial\tilde{D})] \rightarrow \tilde{M}$ to be the parametrization of $\partial\tilde{D}$ by $\sqrt{2k}$ -times the arc length. A simple computation shows that

$$\tilde{S}_k(\tilde{y}_k) = \sqrt{2k} \cdot l(\partial\tilde{D}) + \int_{\tilde{D}} \tilde{\sigma}.$$

Hence, for k sufficiently small $\tilde{S}_k(\tilde{y}_k) < 0$. This shows that $c_+(g, \sigma) > 0$. We now prove that $\tau_+^*(g, \sigma) \geq c_+(g, \sigma)$. Clearly $c(g, \sigma) \geq c_+(g, \sigma)$. Thus, let us show that $\tau_+(g, \sigma) \geq c_+(g, \sigma)$. Suppose $k < c_+(g, \sigma)$. Then there exists a positively oriented simple loop $\tilde{\gamma}$ on the universal cover of M such that $\tilde{S}_k(\tilde{\gamma}) < 0$. Now since M is a surface, $\pi_1(M)$ is *residually finite* (i.e. the intersection of all its normal subgroups with finite index is trivial; see [Hem72] or [Bau62]) and therefore we can find a finite cover $p' : M' \rightarrow M$ such that the projection $\gamma' = p''(\tilde{\gamma})$ to M' is simple, where $p'' : \tilde{M} \rightarrow M'$ (see [CMP04] for a similar argument where the Abelian cover is considered instead). Therefore, γ' bounds an embedded disc D' on M' (actually $D' = p''(\tilde{D})$) and there holds

$$S_k(\gamma') = \tilde{S}_k(\tilde{\gamma}) < 0.$$

From the inequality $\sqrt{2k} \ell \leq kT + \frac{1}{2T} \ell^2$ for all $T > 0$ it follows that

$$\mathcal{T}_k(D') = \sqrt{2k} \cdot l(\gamma') + \int_{D'} \sigma \leq S_k(\gamma') < 0,$$

which shows that $k < \tau_+(g, \sigma)$. □

We can now state Taimanov's main theorem about the existence of global minimizers for \mathcal{T}_k . It is worth to point out that this result does not depend on the genus of the surface and holds more generally for any oscillating non-exact 2-form. For the proof we refer to [Tai92b] for the case $M = S^2$ and to [Tai93] for the general case. The reader can also take a look at [CMP04] for a new proof using methods of geometric measure theory.

Theorem 6.3 (Taimanov, 1992). *Let (M, g) be an oriented surface and let σ be an oscillating non-exact 2-form with positive integral over M . For every $k < \tau_+(g, \sigma)$ there is a smooth positively oriented embedded surface Π which is a global minimizer of \mathcal{T}_k on the space of positively oriented surfaces on a finite cover M' with $\mathcal{T}_k(\Pi) < 0$. Each boundary component of Π is then a simple closed magnetic geodesic for the flow defined by σ on M' .*

Now we show how, in the case $k < \tau_+(g, \sigma)$, from this result we can get the existence of a closed magnetic geodesic which is a local minimizer for S_k .

Lemma 6.4. *Suppose $k < \tau_+(g, \sigma)$. Then, there is a closed magnetic geodesic γ with energy k which is a local minimizer of S_k in Λ . Furthermore, if γ is contractible, we may suppose that $S_k(\gamma) < 0$.*

Proof. If $k < \tau_+(g, \sigma)$, then by Theorem 6.3 we get the existence of a positively oriented embedded surface Π which is a global minimizer of \mathcal{T}_k in the space of positively oriented embedded surfaces on M' and such that $\mathcal{T}_k(\Pi) < 0$. By (10) it follows that Π has non-empty boundary. Each of its boundary components is then a simple magnetic geodesic for the lifted flow. Take a boundary component γ and let y be a simple closed curve which is C^1 -close to γ . Consider the surface Π_y obtained from Π by changing the boundary component γ with y . Since Π is a global minimizer of \mathcal{T}_k among the space of positively oriented embedded surfaces, we get

$$(13) \quad \mathcal{T}_k(\Pi_y) \geq \mathcal{T}_k(\Pi).$$

Observe that

$$\begin{aligned} \mathcal{T}_k(\Pi_y) - \mathcal{T}_k(\Pi) &= \sqrt{2k} \cdot l(\partial\Pi_y) + \int_{\Pi_y} \sigma - \sqrt{2k} \cdot l(\partial\Pi) - \int_{\Pi} \sigma \\ &= \sqrt{2k} \cdot [l(y) - l(\gamma)] + \int_{\Pi_y \setminus \Pi} \sigma. \end{aligned}$$

Hence,

$$\begin{aligned} S_k(y) - S_k(\gamma) &\stackrel{(*)}{\geq} \sqrt{2k} \cdot [l(y) - l(\gamma)] + \int_{C(y)} \sigma - \int_{C(\gamma)} \sigma \\ &\stackrel{(**)}{=} \sqrt{2k} \cdot [l(y) - l(\gamma)] + \int_{\Pi_y \setminus \Pi} \sigma = \mathcal{T}_k(\Pi_y) - \mathcal{T}_k(\Pi), \end{aligned}$$

where in the inequality $(*)$ we have used again the elementary estimate

$$\frac{1}{2T} \ell^2 + kT \geq \sqrt{2k} \ell$$

and the fact that

$$S_k(\gamma) = \sqrt{2k} l(\gamma) + \int_{C(\gamma)} \sigma$$

since γ has constant speed, being a magnetic geodesic. The equality $(**)$ is a consequence of the fact that the form σ is exact on each 2-torus.

Therefore, it follows from (13) that each boundary component of Π is a local minimizer for the functional S_k in $C^1(\mathbb{T}, M') \times (0, +\infty)$. Since the projection map

$$C^1(\mathbb{T}, M') \times (0, +\infty) \longrightarrow C^1(\mathbb{T}, M) \times (0, +\infty)$$

is open, the projection of each boundary component of Π defines a closed magnetic geodesic on M which is a local minimizer for S_k on the space $C^1(\mathbb{T}, M) \times (0, +\infty)$. Notice that these closed magnetic geodesics might not be simple after the projection.

We observe now that (strict) local minimizers in $C^1(\mathbb{T}, M) \times (0, +\infty)$ are also (strict) local minimizers in $H^1(\mathbb{T}, M) \times (0, +\infty) = \Lambda$. This follows from [AMP13, Lemma 2.1], since, as we have seen in Section 4, locally S_k reduces to a classical Lagrangian action functional. Hence, we conclude that γ is a minimizer of S_k on Λ .

From the discussion above we see that the lemma is proven if we can pick γ to be non-contractible. If this is not possible, it means that the boundary components of all global minimizers of \mathcal{T}_k on M' are contractible curves. Pick one of these global minimizers Π with the minimal number, say n , of boundary components; notice that $n \geq 1$ by (10). Recall that in the contractible case we made the natural choice of taking a constant loop as the reference loop in the definition of S_k . Denote by $\gamma_1, \dots, \gamma_n$ the connected components of the boundary of Π parametrized by $\sqrt{2k}$ times the arc-length. Let $\{D_1, \dots, D_n\}$ be the closed embedded discs bounded by the γ_i 's respectively. Clearly there exists an i_* such that one of the two alternative holds:

- (1) $D_{i_*} \subseteq \Pi$ (D_{i_*} is positively oriented), or
- (2) $D_{i_*} \cap \Pi = \gamma_{i_*}$ (D_{i_*} is negatively oriented).

In the first case, $\Pi \setminus D_{i_*}$ is a (possibly empty) positively oriented embedded surface and

$$\mathcal{T}_k(\Pi) = \mathcal{T}_k(\Pi \setminus D_{i_*}) + S_k(\gamma_{i_*}).$$

Since Π is an *absolute* minimizer, we get that $S_k(\gamma_{i_*}) \leq 0$. We claim that, actually, $S_k(\gamma_{i_*}) < 0$. Otherwise, assume by contradiction $S_k(\gamma_{i_*}) = 0$, then $\Pi \setminus D_{i_*}$ is also a global minimizer for \mathcal{T}_k but its boundary has one connected component less than Π , which is a contradiction.

In the second case, let \overline{D}_{i_*} be the disc D_{i_*} taken with the opposite orientation. Then, $\Pi \cup \overline{D}_{i_*}$ is a positively oriented embedded surface and

$$\mathcal{T}_k(\Pi) = \mathcal{T}_k(\Pi \cup \overline{D}_{i_*}) + S_k(\gamma_{i_*}).$$

As before, we conclude that $S_k(\gamma_{i_*}) < 0$.

In both cases, projecting to M if necessary we get the existence of a local minimizer of S_k with negative action and the lemma follows. \square

We conclude this section mentioning the fact that, since M is an orientable surface, a closed curve in M which is a (strict) local minimizer of S_k on Λ remains a (strict) local minimizer also when iterated. Counterexamples to this statement for geodesic flows in dimension bigger than two or on non-orientable surfaces are described in [Hed32] and [KH95, Example 9.7.1], respectively.

Lemma 6.5. *If α is a (strict) local minimizer of S_k on Λ , then for every $n \geq 1$ the n -th iterate α^n is also a (strict) local minimizer of S_k on Λ .*

The proof in [AMP13, Lemma 3.1] for the case of the Lagrangian action functional goes through without any change, thanks to the discussion in Section 4.

7. THE CONTACT PROPERTY

In the next two sections we will complete the proof of the Main Theorem. A decisive step will be to prove that some Palais-Smale sequence (x_n, T_n) has an accumulation point (see Lemma 9.1). As we have seen in Theorem 3.2, the crucial issue is that, in general, there might exist Palais-Smale sequences such that $T_n \rightarrow +\infty$. In the exact case, Contreras showed how this problem is related with the contact geometry of the level set [Con06, Proposition F]. Namely, he proved that every Palais-Smale sequence with energy k has bounded period, provided the energy level $E^{-1}(k)$ is of contact type. Since being of contact type is an open condition, showing that $k \in (0, c(L))$ is an energy value of contact type allows to prove that there are infinitely many magnetic geodesics with *every* fixed energy close to k .

In light of a possible generalization of Contreras' result to our setting, it is natural to try to find out which energy levels are of contact type, when the magnetic form is oscillating. However, so far only few facts are known in this case. The only positive result is Corollary 4.14 in [Ben14] (see also [Pat09]), which asserts that $E^{-1}(k)$ is of positive contact type for all large enough k . On the other hand, obstructions to the contact property are obtained by computing the action of the Liouville measure. Following [Pat09], there is an energy value $c_h(g, \sigma)$ called the *helicity* such that

- the inequality $0 < c_h(g, \sigma) \leq c(g, \sigma)$ holds,
- the action of the Liouville measure on $E^{-1}(k)$ is positive for $k > c_h(g, \sigma)$ and negative for $k < c_h(g, \sigma)$.

Hence, $E^{-1}(k)$ cannot be of *negative* contact type for $k \geq c_h(g, \sigma)$ and it cannot be of *positive* contact type for $k \leq c_h(g, \sigma)$.

Since we are looking at the energy range $(0, \tau_+^*(g, \sigma))$, we could hope that low energy levels are of negative contact type, as happens when σ is symplectic [Pat09, Ben14]. However, the next proposition shows that, when the magnetic form is oscillating, low energy levels are never of *negative* contact type.

Proposition 7.1. *Let (M, g, σ) be an oscillating magnetic system on an orientable Riemannian surface with genus at least two. For every small enough $k > 0$ the energy level $E^{-1}(k)$ is not of negative contact type. As a corollary, $E^{-1}(k)$ is not of contact type.*

Proof. Define $\bar{\sigma} := K\mu \in \Omega^2(M)$, where K is the Gaussian curvature and μ is the area form of g . There exists $\beta \in \Omega^1(M)$ such that $\sigma = a_\sigma \bar{\sigma} + d\beta$, where

$$a_\sigma := \frac{\int_M \sigma}{\int_M \bar{\sigma}} = \frac{\int_M \sigma}{2\pi \chi(M)} < 0$$

by the Gauss-Bonnet Theorem. Following the computation in [Ben14, Proposition 4.5], the action of a null-homologous φ_t -invariant measure ζ on $E^{-1}(k)$ is given by

$$\mathcal{A}_k(\zeta) = \int_{E^{-1}(k)} \left[2k + \beta_x(v) + a_\sigma f(x) \right] d\zeta,$$

where $f : M \rightarrow \mathbb{R}$ is defined by the equation $\sigma = f\mu$. Since σ is oscillating, $\min f < 0$.

The energy level $E^{-1}(k)$ is not of negative contact type, provided there exists a null-homologous φ_t -invariant measure ζ on $E^{-1}(k)$ with $\mathcal{A}_k(\zeta) \geq 0$. Denote by

$$C_c^f := \{x \in M \mid f(x) \leq c\}$$

the closed sublevels of f and let $b \in (\min f, 0)$ be a regular value of f . Construct now a function $f_* : M \rightarrow (-\infty, 0)$ such that f_* coincides with f on the sublevel C_b^f and $C_b^{f_*} = C_b^f$. Since f_* is a negative function, we can define the Ginzburg action functional $G_k^{f_*} : E^{-1}(k) \rightarrow \mathbb{R}$ for all sufficiently small $k > 0$ (see [Gin87] for further details). This functional has the property that its critical points are the support of those periodic orbits γ such that

$$(14) \quad l(\pi(\gamma)) \leq C\sqrt{2k},$$

for some $C > 0$ independent of k . As is proved in [Ben14] we can expand this functional in the parameter k around zero getting

$$G_k^{f_*}(x, v) = 2\pi - \frac{2\pi}{f_*(x)}k + o(k).$$

For every k let now (x_k, v_k) be an absolute maximizer for $G_k^{f_*}$. By the expansion above for every $\varepsilon > 0$ there exists k_ε such that

$$d(x_k, C_{\min f_*}^{f_*}) < \varepsilon, \quad \text{for } k < k_\varepsilon.$$

Call γ_k the periodic orbit through (x_k, v_k) . By Inequality (14), there is $k'_\varepsilon \leq k_\varepsilon$ such that $\pi(\gamma_k) \subset C_b^{f_*} = C_b^f$ if $k < k'_\varepsilon$. If ζ_k is the invariant measure supported on γ_k , then

$$\mathcal{A}_k(\zeta_k) = \int_{E^{-1}(k)} \left[2k + \beta_x(v) + a_\sigma f(x) \right] d\zeta_k \geq 2k - \|\beta\|\sqrt{2k} + a_\sigma \cdot b.$$

Since $a_\sigma \cdot b > 0$, $\mathcal{A}_k(\zeta_k) \geq 0$ for k sufficiently small. □

In view of the previous proposition, we will really need Struwe's argument, contained in Lemma 9.1, to prove the existence of converging Palais-Smale sequences (even if only on *almost every* energy level).

8. THE MINIMAX VALUES

In this section we follow [AMMP14, Section 3] and define the minimax values which allow us to prove the existence of infinitely many closed magnetic geodesics on almost every energy level below $\tau_+^*(g, \sigma)$. For the reader's convenience we keep the same notation as in the main reference.

For any $k \in (0, \tau_+^*(g, \sigma)) \subseteq (0, c(g, \sigma))$, let $\alpha_k \in \Lambda$ be a local minimizer of S_k , whose existence is guaranteed by Lemma 6.4. Recall that if α_k is contractible we may suppose $S_k(\alpha_k) < 0$. For the rest of the paper let us fix $k_* \in (0, \tau_+^*(g, \sigma))$. Without loss of generality we may assume that α_{k_*} is a strict local minimizer for S_{k_*} , as otherwise there trivially exist infinitely many geometrically distinct closed magnetic geodesics with energy k_* . By Lemma 6.5 all iterates $\alpha_{k_*}^n$ are also strict local minimizers. Denote by ν_* the free homotopy class of α_{k_*} . Thanks to Lemma 2.6 and to the fact that on a surface of genus at least two a free-homotopy class is either the trivial class or has infinite order, we can take S_k to be \mathbb{N} -equivariant on $\bigcup_{n \in \mathbb{N}} \Lambda_{\nu_*}^n$.

Since k_* is strictly smaller than $c(g, \sigma)$, by Lemma 2.3 the functional S_{k_*} is unbounded from below on any connected component of Λ and hence there exists an element $\mu \in \Lambda_{\nu_*}$ such that

$$S_{k_*}(\mu) < S_{k_*}(\alpha_{k_*}).$$

Thanks to Lemma 3.3 we can find a bounded open neighborhood $\mathcal{U} \subset \Lambda_{\nu_*}$ of $\mathbb{T} \cdot \alpha_{k_*}$ such that

- its closure intersects the critical set of S_{k_*} only in $\mathbb{T} \cdot \alpha_{k_*}$,
- the inequality $\inf_{\partial \mathcal{U}} S_{k_*} > S_{k_*}(\alpha_{k_*})$ holds,
- the period T is bounded away from zero for all $(x, T) \in \mathcal{U}$.

For every $k \in (0, \tau_+^*(g, \sigma))$ let M_k be the closure of the set of local minimizers of S_k contained in \mathcal{U} . All the elements of M_k are critical points for S_k and they are also strict local minimizers, provided M_k is a finite union of critical circles. The proof of the following lemma is analogous to the one of [AMMP14, Lemma 3.1].

Lemma 8.1. *There exists a closed interval $J = J(k_*) \subset (0, \tau_+^*(g, \sigma))$ whose interior contains k_* and which has the following properties:*

- (1) *For every $k \in J$ the set M_k is a non-empty compact set.*
- (2) *For every $k \in J$ there holds*

$$S_k(\mu) < \min_{M_k} S_k.$$

- (3) *For every $k \in J$ there holds*

$$\sup_{k' \in J} \max_{M_{k'}} S_k < \inf_{\partial \mathcal{U}} S_k;$$

if α_{k_} is contractible, there also holds $\sup_{k' \in J} \max_{M_{k'}} S_k < 0$.*

For every $n \in \mathbb{N}$ and every $k \in J$ we define

$$\mathcal{P}_n(k) := \left\{ u \in C^0([0, 1], \Lambda) \mid u(0) \in \psi^n(M_k), u(1) = \mu^n \right\}$$

as the set of continuous paths in Λ which join the n -th iterate of some element in M_k with the n -th iterate of μ . They yield the minimax functions

$$c_n : J \longrightarrow \mathbb{R}, \quad c_n(k) := \inf_{u \in \mathcal{P}_n(k)} \max_{s \in [0,1]} S_k(u(s)).$$

Applying point (2) of the previous lemma, we have

$$(15) \quad c_n(k) \geq \min_{\psi^n(M_k)} S_k = n \cdot \min_{M_k} S_k > n \cdot S_k(\mu).$$

As already discussed in the introduction, in order to overcome the possible lack of the Palais-Smale condition we will use the fact that each of the minimax functions c_n is monotone. Strictly speaking, we use the fact that a monotone function is differentiable at almost every point to construct bounded Palais-Smale sequences at almost every level k (namely at each level where all the functions c_n are differentiable). Unfortunately, this monotonicity property does not follow directly from the monotonicity of the functional S_k , since the minimax class $\mathcal{P}_n(k)$ does depend on k . However, monotonicity can be shown exactly as done in [AMMP14] for the exact case and, hence, we omit the proof.

Lemma 8.2. *For every $n \in \mathbb{N}$, c_n is monotonically increasing on J .*

If the strict local minimizer α_k is contractible, we need to know more about the behavior of the minimax functions c_n . The next lemma is a simple generalization of [AMMP14, Lemma 3.4] and is based on the so-called “technique of pulling one loop at a time” introduced first by Bangert [Ban80]. Observe that we can run the same proof as in [AMP13, Lemma 2.5] since

$$S_k|_{\Lambda_0} \circ p = \tilde{S}_k,$$

where $p : \tilde{\Lambda} \rightarrow \Lambda$ denotes the projection map. Indeed, no assumption on the compactness of M is required in that argument.

Lemma 8.3. *Let $K_0, K_1 \subseteq \Lambda_0$ be compact sets and let*

$$\mathcal{R}_n := \left\{ u \in C^0([0,1], \Lambda_0) \mid u(0) \in \psi^n(K_0), u(1) \in \psi^n(K_1) \right\}.$$

Fix a number k and set

$$c_n := \inf_{u \in \mathcal{R}_n} \max_{s \in [0,1]} S_k(u(s)).$$

Then, there exists a number $A \in \mathbb{R}$ such that

$$c_n \leq n \cdot \max_{K_0 \cup K_1} S_k + A, \quad \forall n \in \mathbb{N}.$$

We apply now this lemma to $\hat{k} = \max J$, $K_0 = M_{\hat{k}}$ and $K_1 = \{\mu\}$ and use the fact that $S_{\hat{k}}(\mu)$ and $\max_{M_{\hat{k}}} S_{\hat{k}}$ are negative (see Lemma 8.1) to obtain

$$c_n(\hat{k}) \longrightarrow -\infty \quad \text{for } n \longrightarrow +\infty.$$

In particular, since each c_n is monotonically increasing in k , we get

$$(16) \quad \lim_{n \rightarrow +\infty} c_n = -\infty \quad \text{uniformly on } J.$$

In the non-contractible case, a simple topological argument will play the same role as (16) in the proof of the Main Theorem (see the last section for further details).

9. PROOF OF THE MAIN THEOREM

Here we complete the proof of the Main Theorem following Section 3.3 in [AMMP14]. By the discussion in the previous sections this will be an easy consequence of Theorem 9.2 below.

We saw in Theorem 3.2 that Palais-Smale sequences for the functional S_k at level c in the same free homotopy class and with period bounded from above have a limit point, provided:

- (1) They are not contractible, or
- (2) They are contractible and $c \neq 0$.

The next lemma shows that sequences satisfying such bound exist on almost every energy level close to k_* . The original argument is due to Struwe [Str90] (see also [Con06, Proposition 7.1]). The formulation we give here is a generalization of Lemma 3.5 in [AMMP14] to the weakly exact case and the proof contained therein goes through word by word. Let $c_n : J \rightarrow \mathbb{R}$ be the sequence of minimax functions defined in the previous section. In case the strict local minimizer α_{k_*} is contractible, we know by (16) that there exists $n_0 \in \mathbb{N}$ such that all c_n 's are negative for $n \geq n_0$.

Lemma 9.1. *Let k be an interior point of J at which c_n is differentiable and such that the set M_k is a finite union of critical circles. If α_{k_*} is contractible suppose in addition that $n \geq n_0$. Then, for every open neighborhood \mathcal{V} of*

$$\text{crit } S_k \cap \{S_k = c_n(k)\},$$

there exists an element v of $\mathcal{P}_n(k)$ such that $S_k(v(0)) < c_n(k)$ and

$$v([0, 1]) \subset \{S_k < c_n(k)\} \cup \mathcal{V}.$$

In particular $c_n(k)$ is a critical value of $S_k|_{\Lambda_{\nu_^n}}$.*

Theorem 9.2. *If the local minimizer α_{k_*} is strict, then the energy value k_* has a neighborhood $J \subseteq (0, \tau_+^*(g, \sigma))$ such that for almost every $k \in J$ the energy level $E^{-1}(k)$ has infinitely many periodic orbits freely homotopic to some iteration of α_{k_*} .*

Proof. Let $J = J(k_*)$ and $c_n : J \rightarrow \mathbb{R}$ be as above. Since the countably many functions c_n are monotone, it follows from the Lebesgue Theorem that the set of points J' at which all the c_n are differentiable has full measure in J . We shall prove that for every $k \in J'$ the energy level $E^{-1}(k)$ has infinitely many geometrically distinct periodic orbits.

Thus, pick $k \in J'$. If M_k consists of infinitely many critical circles, then there is nothing to prove. Therefore, we may suppose that M_k consists of only finitely many critical circles, which are local minimizers. Assume by contradiction that $E^{-1}(k)$ has only finitely many periodic orbits homotopic to some iteration of α_{k_*} . Then, the critical set of S_k restricted to $\bigcup_{n \in \mathbb{N}} \Lambda_{\nu_*^n}$ consists of finitely many critical circles

$$\mathbb{T} \cdot \gamma_1, \mathbb{T} \cdot \gamma_2, \dots, \mathbb{T} \cdot \gamma_l,$$

together with their iterates $\mathbb{T} \cdot \gamma_j^n$, for $1 \leq j \leq l$ and for every $n \in \mathbb{N}$. By Theorem 5.1 we can find a natural number n_* such that the following is true: for every $n \geq n_*$ and for every $1 \leq j \leq l$ there exists a neighborhood $\mathcal{W}_{j,n}$ of $\mathbb{T} \cdot \gamma_j^n$ such that any two points in $\{S_k < S_k(\gamma_j^n)\}$ which can be connected within

$$\{S_k < S_k(\gamma_j^n)\} \cup \mathcal{W}_{j,n}$$

can be also connected in $\{S_k < S_k(\gamma_j^n)\}$. Moreover, the sets $\mathcal{W}_{j,n}$ can be chosen to be so small that their closures are pairwise disjoint. If α_{k_*} is not contractible we can find $n \in \mathbb{N}$

such that $\alpha_{k_*}^n$ is not freely homotopic to γ_j^m , for every $1 \leq j \leq l$ and $1 \leq m \leq n_* - 1$. If α_{k_*} is contractible, set

$$a := \min_{1 \leq j \leq l} S_k(\gamma_j^{n_*-1}).$$

By (16) we can find a natural number $n \geq n_0$ such that $c_n(k) < a$. In any case, Lemma 9.1 implies that the value $c_n(k)$ is a critical value of S_k restricted to $\Lambda_{\nu_*^n}$. Thanks to our finiteness assumption

$$\text{crit } S_k|_{\Lambda_{\nu_*^n}} \cap \{S_k = c_n(k)\} = \mathbb{T} \cdot \gamma_{j_1}^{m_1} \cup \dots \cup \mathbb{T} \cdot \gamma_{j_r}^{m_r}$$

for some non-empty set $\{j_1, \dots, j_r\} \subseteq \{1, \dots, l\}$ and some positive integers m_1, \dots, m_r . By our choice of n , all the m_i 's are at least n_* . Now we apply Lemma 9.1 with

$$\mathcal{V} := \mathcal{W}_{j_1, m_1} \cup \dots \cup \mathcal{W}_{j_r, m_r}$$

and we obtain a path $v \in \mathcal{P}_n(k)$ with image contained in

$$\{S_k < c_n(k)\} \cup \mathcal{V}$$

and such that $S_k(v(0)) < c_n(k)$. Since also $S_k(v(1)) = S_k(\mu^n) < c_n(k)$, and since the sets \mathcal{W}_{j_i, m_i} have pairwise disjoint closures, the path v is the concatenation of finitely many paths, each of which has end-points in $\{S_k < c_n(k)\}$ and is contained in

$$\{S_k < c_n(k)\} \cup \mathcal{W}_{j_i, m_i},$$

for some $i \in \{1, \dots, r\}$. By the property of the sets \mathcal{W}_{j_i, m_i} stated above, the end-points of each of these paths can be joined by paths w in $\{S_k < c_n(k)\}$ and hence by concatenating the w 's we get a path in $\{S_k < c_n(k)\}$ which joins $u(0)$ to $u(1)$. Since such a path belongs to $\mathcal{P}_n(k)$, this contradicts the definition of $c_n(k)$. \square

REFERENCES

- [Abb13] A. Abbondandolo, *Lectures on the free period Lagrangian action functional*, J. Fixed Point Theory Appl. **13** (2013), no. 2, 397–430.
- [AMMP14] A. Abbondandolo, L. Macarini, M. Mazzucchelli, and G. P. Paternain, *Infinitely many periodic orbits of exact magnetic flows on surfaces for almost every subcritical energy level*, preprint, arXiv:1404.7641, 2014.
- [AMP13] A. Abbondandolo, L. Macarini, and G. P. Paternain, *On the existence of three closed magnetic geodesics for subcritical energies*, to appear in Comm. Math. Helv., arXiv:1305.1871, 2013.
- [Ban80] V. Bangert, *Closed geodesics on complete surfaces*, Math. Ann. **251** (1980), no. 1, 83–96.
- [Bau62] G. Baumslag, *On generalised free products*, Math. Z. **78** (1962), 423–438.
- [Ben14] G. Benedetti, *The contact property for magnetic flows on surfaces*, Ph.D. thesis, University of Cambridge, 2014.
- [BP02] K. Burns and G. P. Paternain, *Anosov magnetic flows, critical values and topological entropy*, Nonlinearity **15** (2002), no. 2, 281–314.
- [CI99] G. Contreras and R. Iturriaga, *Global minimizers of autonomous Lagrangians*, 22^o Colóquio Brasileiro de Matemática. [22nd Brazilian Mathematics Colloquium], Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1999.
- [CIPP00] G. Contreras, R. Iturriaga, G. P. Paternain, and M. Paternain, *The Palais-Smale condition and Mañé's critical values*, Ann. Henri Poincaré **1** (2000), no. 4, 655–684.
- [CMP04] G. Contreras, L. Macarini, and G. P. Paternain, *Periodic orbits for exact magnetic flows on surfaces*, Int. Math. Res. Not. (2004), no. 8, 361–387.
- [Con06] G. Contreras, *The Palais-Smale condition on contact type energy levels for convex Lagrangian systems*, Calc. Var. Partial Differential Equations **27** (2006), no. 3, 321–395.
- [GG09] V. Ginzburg and B. Z. Gürel, *Periodic orbits of twisted geodesic flows and the Weinstein-Moser theorem*, Comment. Math. Helv. **84** (2009), no. 4, 865–907.
- [GGM14] V. L. Ginzburg, B. Gürel, and L. Macarini, *On the Conley conjecture for Reeb flows*, preprint, arXiv:1407.1773v1, 2014.
- [Gin87] V. L. Ginzburg, *New generalizations of Poincaré's geometric theorem*, Funktsional. Anal. i Prilozhen. **21** (1987), no. 2, 16–22, 96.

- [GM69] D. Gromoll and W. Meyer, *Periodic geodesics on compact riemannian manifolds*, J. Differential Geometry **3** (1969), 493–510.
- [Hed32] G. A. Hedlund, *Geodesics on a two-dimensional Riemannian manifold with periodic coefficients*, Ann. of Math. (2) **33** (1932), no. 4, 719–739.
- [Hem72] J. Hempel, *Residual finiteness of surface groups*, Proc. Amer. Math. Soc. **32** (1972), 323.
- [Ker99] E. Kerman, *Periodic orbits of Hamiltonian flows near symplectic critical submanifolds*, Internat. Math. Res. Notices (1999), no. 17, 953–969.
- [KH95] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995, With a supplementary chapter by Katok and Leonardo Mendoza.
- [Mer10] W. J. Merry, *Closed orbits of a charge in a weakly exact magnetic field*, Pacific J. Math. **247** (2010), no. 1, 189–212.
- [Pat06] G. P. Paternain, *Magnetic rigidity of horocycle flows*, Pacific J. Math. **225** (2006), no. 2, 301–323.
- [Pat09] ———, *Helicity and the Mañé critical value*, Algebr. Geom. Topol. **9** (2009), no. 3, 1413–1422.
- [Sch06] F. Schlenk, *Application of Hofer’s geometry to Hamiltonian dynamics*, Comment. Math. Helv. **81** (2006), no. 1, 105121.
- [Str90] M. Struwe, *Existence of periodic solutions of Hamiltonian systems on almost every energy surface*, Bol. Soc. Brasil. Mat. (N.S.) **20** (1990), no. 2, 49–58.
- [Tai92a] I. A. Taĭmanov, *Closed extremals on two-dimensional manifolds*, Russian Math. Surveys **47** (1992), no. 2, 163–211.
- [Tai92b] ———, *Non-self-intersecting closed extremals of multivalued or not-everywhere-positive functionals*, Math. USSR-Iz. **38** (1992), no. 2, 359–374.
- [Tai93] ———, *Closed non-self-intersecting extremals of multivalued functionals*, Siberian Math. J. **33** (1993), no. 4, 686–692.
- [Ush09] M. Usher, *Floer homology in disk bundles and symplectically twisted geodesic flows*, J. Mod. Dyn. **3** (2009), no. 1, 61–101.

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